Section 11.3

Taylor Series

We now have a way to represent functions as series and an approximation of differentiable function with polynomials. It is time to connect the two concepts.

Definition 11.3.1. Taylor/Maclaurin Series

Suppose the function f has derivatives of all orders on an interval centered at a. The *Taylor series for f centered at a* is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A Taylor series centered at 0 is called a *Maclaurin series.*

Example 11.3.2. Find the Maclaurin series for $f(x) = \cos x$.

Example 11.3.3. Find the Taylor series of $f(x) = \dfrac{1}{1-x}$ centered at $a = 5$.

Example 11.3.4. Find the Maclaurin series and its interval of convergence for $f(x) = e^x$.

Binomial Series

The Taylor series for functions of the form $(1+x)^p$ are special. When p is a positive integer

$$(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p}x^p$$

Definition 11.3.5. Binomial Coefficients

For real numbers p and integers $k \geq 1$,

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \quad \binom{p}{0} = 1$$

Theorem 11.3.6. Binomial Series

For real numbers $p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at $a = 0$ is the *binomial series*

$$\sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

The series converges for $|x| < 1$. If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Example 11.3.7. Find the Maclaurin series for $f(x) = \sqrt{1+x}$.

Example 11.3.8. Find the Maclaurin series for $g(x) = \sqrt[3]{c+x}$.

Convergence of Taylor Series

While it is obvious that the Taylor Series we have constructed are power series, it is not obvious that they actually converge to their respective functions. This is handled by Taylor's Theorem.

Theorem 11.3.9. Convergence of Taylor Series

Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{x\to\infty} R_n(x) = 0$ for all x in I, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x.

Example 11.3.10. Show that the Maclaurin series of $f(x) = e^x$ converges to $f(x)$ for $-\infty < x < \infty$.

Useful Taylor Series

1. $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$
2. $\dfrac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$
3. $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$
4. $\sin x = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}$
5. $\cos x = \sum_{k=0}^{\infty} \dfrac{(-1)^{k+1} x^{2k}}{(2k)!}$
6. $\ln(1+x) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1} x^k}{k}$
7. $-\ln(1-x) = \sum_{k=1}^{\infty} \dfrac{x^k}{k}$
8. $\arctan x = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{2k+1}$
9. $\sinh x = \sum_{k=0}^{\infty} \dfrac{x^{2k+1}}{(2k+1)!}$
10. $\cosh x = \sum_{k=0}^{\infty} \dfrac{x^{2k}}{(2k)!}$
11. $(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k$

Example 11.3.11. Find the Taylor Series for the following

1. $f(x) = (1 + x^2)^{-1}$ at $a = 0$

2. $f(x) = \dfrac{1}{x^2}$ at $a = 1$

3. $f(x) = x \sin x$ at $a = 0$

4. $f(x) = \ln x$ at $a = 1$

5. $f(x) = x \ln x - x + 1$ at $a = 1$